REFERENCES

1. Moser, J. K., Lectures on Hamiltonian Systems. Mem. Amer. Math. Soc., № 81, 1968.
2. Birkhoff, G. D., Dynamical Systems. New York, Amer. Math. Soc., 1927.
3. Markeev, A. P., Resonance effects and stability of steady-state rotations of an artificial satellite. Kosmicheskie Issledovaniia, Vol. 5, № 3, 1967.
4. Markeev, A. P., On the stability of the triangular libration points in the elliptic restricted three-body problem. PMM Vol. 34, № 2, 1970.
5. Khazin, L. G., On the stability of Hamiltonian systems in the presence of resonances. PMM Vol. 35, № 3, 1971.
6. Sigel, C. L., Vorlesungen über Himmelsmechanik. Berlin, Springer Verlag, 1957. (Russian translation: Lectures on Celestial Mechanics. Moscow, Izd. Inostr. Lit., 1959).
7. Nemytskii, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations. Moscow-Leningrad, Gostekhizdat, 1949. (English translation: Qualitative Theory of Differential Equations. Princeton, Princeton Univ. Press, 1960).
8. Demidovich, B. P., Lectures on the Mathematical Theory of Stability. Moscow, "Nauka", 1967.
9. Daletskii, Iu. L. and Krein, M. G., Stability of Solutions of Differential Equations in Banach Space. Moscow, "Nauka", 1970.
10. Stepanov, V. V., Course of Differential Equations. Moscow, Fizmatgiz, 1959.
11. Sokol'skii, A. G., On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies. PMM Vol. 38, № 5, 1974.
12. Sokol'skii, A. G., On the stability of the Lagrange solutions of the restricted three-body problem with a critical mass ratio. PMM Vol. 39, № 2, 1975.

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SYNTHESIS OF DISCRETE VIBRATIONAL SYSTEMS WITH MAXIMALLY COMPRESSED SPECTRUM

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We propose a synthesis method for the parameter group of discrete vibrational systems, ensuring the maximal compression of the natural frequency spectrum. We give a method for solving two problems: (1) for a specified spectrum and definite part of the parameters find the values of the remaining parameters so that the lowest frequency would occupy the given position on the number axis and that the ratio of the highest frequency to the lowest would be minimal; (2) for a specified vibrational system obtain a system with maximally compressed spectrum at the expense of optimal vibration of a definite group of parameters.

We examine discrete vibarational systems with m degrees of freedom, whose amplitude equations are described by a generalized amplitude equation [1] of the form

$$Du\mathbf{x} = \mu A\mathbf{x} \tag{1}$$

Here $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ is the vector of first physical parameters, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is the oscillation vector of generalized form, μ is an eigenvalue of the generalized amplitude equation (the square of the generalized natural frequency of the vibrational system), D is a linear operator taking an arbitrary k-dimensional vector $\mathbf{b}=(b_1, b_2, \ldots, b_k)$ into a kth-order diagonal one: $D\mathbf{b} = \mathrm{diag}\,(b_1, b_2, \ldots, b_k)$, $A = \| A_{ij} \|_1^m$ is a matrix determined by the structure of the vibrational system and by the vector of second physical parameters. We assume that matrix A possesses the following properties:

1°. $A_{ij} = A_{ji}, \quad i, j = 1, 2, \ldots, m$

2°. $A_{ij} \geqslant 0, \quad i, j = 1, 2, \ldots, m$

3°. An integer k exists such that all elements of matrix A^k are strictly positive.

4°. $(A\tau, \tau) > 0$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ is an arbitrary m-dimensional vector.

5°. There exists a vector of sign alternations [1]

$$\mathbf{v} = ((-1)^{p_1}, (-1)^{p_2}, \ldots, (-1)^{p_m})$$

where $(p_i)_1^{m}$ is some collection of primes, such that the matrix $D\mathbf{v}A^{-1}D\mathbf{v}$ possesses the above-listed properties of matrix A.

It can be shown, using Perron's theorem [2], that the smallest μ_1 and the largest μ_m eigenvalues of Eq. (1) are positive and are the simple roots of the characteristic equation $| D\mathbf{u} - \mu A | = 0$, while the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_m$ of Eq. (1) corresponding to them satisfy the inequalities

$$\mathbf{x}_1 > 0 \tag{2}$$

$$D\mathbf{v}\mathbf{x}_m > 0 \tag{3}$$

It should be noted that in mechanical systems the group of first physical parameters can be made up either from the rigidity (pliability) of the eleastic elements or of the time lag (mobility) of the inertial elements of the vibrational system. The group of second physical parameters is here made up from the time lag (mobility) or the rigidity (pliability), respectively. Here, by the mobility of an inertial element we mean a quantity inverse to its time lag. For the vibrational systems being examined, a number of problems have been solved of synthesizing their parameters under constraints imposed both on the spectrum of the vibrational system as well as on the values of the parameters [3–5]. Below we present a solution of the problems of selecting the first physical parameters of a vibrational system, ensuring a maximally compressed spectrum.

Problem 1. Given the structure of a vibrational system and the vector of second physical parameters. Obtain the vector of first physical parameters, for which the smallest generalized natural frequency occupies a specified position on the number axis and the ratio of the largest frequency to the smallest is minimal.

Peoblem 2. Given the original vibrational system. Obtain, by a minimal change in the vector of first physical parameters, a system having a maximally compressed spectrum.

The following theorem shows that Problem 1 is the main one, since its solution determines the solution of Problem 2.

Theorem 1. An increase of all the coordinates of vector $\mathbf{u}$ by ρ times ($\rho > 0$) does not change the natural forms of the generalized amplitude equation, but leads only to a proportional increase of all the eigenvalues by ρ times.

From Theorem 1 it follows that the first eigenvalue μ_1 can be led by an appropriate proportional change of all the coordinates of the vector of first physical parameters, to a specified position on the number axis without changing the ratio μ_m / μ_1. Thus, if Problems 1 and 2 have been solved for vibrational systems of like structure and like vector of second physical parameters, then their solutions are collinear

$$\mathbf{u}_r = \rho \mathbf{u}_\alpha, \quad \rho > 0$$

Here $\mathbf{u}_r$ is a solution of Problem 2 when vector $\mathbf{r}$ is the original vector of first physical parameters, $\mathbf{u}_\alpha$ is a solution of Problem 1 when $\mu_1 = \alpha$. The collinearity factor ρ is easily determined from the condition that the length of the discrepancy vector $(\mathbf{u}_r - \mathbf{r})$ be a minimum

$$\rho = (\mathbf{r}, \mathbf{u}_\alpha) / (\mathbf{u}_\alpha, \mathbf{u}_\alpha)$$

We proceed to the solution of Problem 1. Having fixed the structure and the vector of second physical parameters of a vibrational system, we change the vector $\mathbf{u}$. The eigenvalues of Eq. (1) will be changed here. We multiply both sides of Eq. (1) scalarly by $\mathbf{x}$ and we find

$$\mu(\mathbf{u}) = (D\mathbf{u}\mathbf{x}(\mathbf{u}), \mathbf{x}(\mathbf{u})) / (A\mathbf{x}(\mathbf{u}), \mathbf{x}(\mathbf{u})) \quad (4)$$

Vector $\mathbf{u}$ can be changed such that one of the eigenvalues of Eq. (1) retains its own value, equal to α ($\alpha > 0$). The set

$$L_\alpha = \{\mathbf{u} \mid \mu(\mathbf{u}) = \alpha, \ \mathbf{u} > 0\}$$

is called the complete equifrequency surface. We note the following interesting peculiarity of the complete equifrequency surface. The vector defined by the relation

$$\mathbf{n} = D\mathbf{x}_e\mathbf{x}_e \quad (5)$$

where $\mathbf{x}_e$ is the natural form of the equation $D\mathbf{u}_e\mathbf{x}_e = \alpha A\mathbf{x}_e$, is normal to surface L_α at the point $\mathbf{u} = \mathbf{u}_e$. In fact, L_α is the level surface of the function $\mu = \mu(\mathbf{u})$, $\mathbf{u} > 0$; therefore, the normal drawn at an arbitrary point $\mathbf{u} \in L_\alpha$ is collinear to the gradient $\nabla\mu(\mathbf{u})$ at the given point. From Eqs. (1) and (4) we obtain

$$d\mu = (D\mathbf{x}\mathbf{x}, d\mathbf{u})/(A\mathbf{x}, \mathbf{x}) \quad (6)$$

Then the gradient's value at point $\mathbf{u} = \mathbf{u}_e$ is

$$\nabla\mu(\mathbf{u}_e) = D\mathbf{x}_e\mathbf{x}_e/(A\mathbf{x}_e, \mathbf{x}_e)$$

We obtain relation (5) by setting the collinearity coefficient equal to $(A\mathbf{x}_e, \mathbf{x}_e)$. The complete equifrequency surface is a generalization of all equifrequency surfaces corresponding to different numbers of eigenvalues

$$L_\alpha = \bigcup_{k=1}^{m} L_{\alpha,k}, \quad L_{\alpha,k} = \{\mathbf{u} \mid \mu_k(\mathbf{u}) = \alpha, \ \mathbf{u} > 0\}$$

Theorem 2. The equifrequency surface $L_{\alpha,1}$ is defined by the parametric equation

$$\mathbf{u} = \alpha D\mathbf{z}^{-1}A\mathbf{z} \quad (7)$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ ranges over the orthant $\mathbf{z} > 0$. The vector $D\mathbf{z}\mathbf{z}$ is normal to the given surface.

The theorem's validity follows from Eq. (1) and relations (2) and (5). The vector $\mathbf{u}_\alpha$, being a solution of Problem 1, belongs to $L_{\alpha,1}$ and satisfies the equality

$$\mu_m(\mathbf{u}_\alpha) = \min_{(\mathbf{u} \in L_{\alpha,1})} \mu_m(\mathbf{u}) \tag{8}$$

Theorem 2 allows us to pass from the problem of finding the conditional extremum (8) to the problem of finding the usual extremum

$$\mu_m(\mathbf{u}_\alpha) = \min_{(\mathbf{z}>0)} \mu_m(\mathbf{u}(\mathbf{z}))$$

The necessary parameters of the vibrational system at the stationary point of function $\mu_m = \mu_m(\mathbf{z})$ are denoted by $\mathbf{u}^\circ, \mathbf{x}_m{}^\circ, \mathbf{z}^\circ, \mu_m{}^\circ$. These parameters are connected by the relation

$$\nabla \mu_m(\mathbf{z}^\circ) = 0 \tag{9}$$

The equality

$$d\mathbf{u} = D\mathbf{z}^{-1}[\alpha A - D\mathbf{u}]\, d\mathbf{z} \tag{10}$$

follows from (7) for a point sliding along the surface $L_{\alpha,1}$. Using relations (6) and (10), we obtain

$$d\mu_m = ([\alpha A - D\mathbf{u}]\, D\mathbf{z}^{-1} D\mathbf{x}_m \mathbf{x}_m, d\mathbf{z}) / (A\mathbf{x}_m, \mathbf{x}_m)$$

Thus,

$$\nabla \mu_m(\mathbf{z}) = [\alpha A - D\mathbf{u}]\, D\mathbf{z}^{-1} D\mathbf{x}_m \mathbf{x}_m / (A\mathbf{x}_m, \mathbf{x}_m)$$

and Eq. (9) is transformed to the form

$$[\alpha A - D\mathbf{u}^\circ]\, \mathbf{y}^\circ = 0, \quad \mathbf{y}^\circ = D\mathbf{z}^{\circ -1} D\overset{\circ}{\mathbf{x}}_m \overset{\circ}{\mathbf{x}}_m \tag{11}$$

Equality (11) can be achieved only on a vector $\mathbf{y}^\circ$ collinear to $\mathbf{z}^\circ$. Setting $\mathbf{y}^\circ = \mathbf{z}^\circ$, we obtain the dependency between the first and the m-th forms of the oscillations at the stationary point

$$D\mathbf{z}^\circ \mathbf{z}^\circ = D\overset{\circ}{\mathbf{x}}_m \overset{\circ}{\mathbf{x}}_m \tag{12}$$

Inequality (3) allows us to transform relation (12) to

$$\overset{\circ}{\mathbf{x}}_m = D\mathbf{v}\mathbf{z}^\circ \tag{13}$$

It can be shown that the function $\mu_m = \mu_m(\mathbf{z})$ reaches the greatest lower bound at the stationary point $\mathbf{z}^\circ$. Thus, $\mathbf{u}_\alpha = \mathbf{u}^\circ$. Equality (12) denotes the tangency of the equifrequency surfaces $L_{\alpha,1}$ and $L_{\beta,m}$ at the point $\mathbf{u} = \mathbf{u}^\circ$, where $\beta = \mu_m(\mathbf{z}^\circ)$. Therefore, point $\mathbf{u}^\circ$ satisfies the equations

$$D\mathbf{u}^\circ \mathbf{z}^\circ = \alpha A \mathbf{z}^\circ, \quad D\mathbf{u}^\circ \overset{\circ}{\mathbf{x}}_m = \beta A \mathbf{x}^\circ$$

Using these equations and relation (13), we obtain the equation for finding $\mathbf{z}^\circ$ and β

$$\alpha A \mathbf{z}^\circ = \beta D\mathbf{v} A D\mathbf{v} \mathbf{z}^\circ, \quad \mathbf{z}^\circ > 0 \tag{14}$$

Let $B = D\mathbf{v} A^{-1} D\mathbf{v} A$, then one of the solutions of the equation

$$B\mathbf{z} = \lambda \mathbf{z} \tag{15}$$

determines a solution of Eq. (14).

We note the following properties of matrix B, following from its definition.

1) Matrix B is similar to a symmetric matrix. In fact, $B = A^{-1/2}[A^{1/2} D\mathbf{v} A^{-1} D\mathbf{v} A^{1/2}]\, A^{1/2}$.

2) The elements of matrix B are nonnegative, but an integer k exists such that all elements of matrix B^k are strictly positive. Indeed, matrix B is the product of two matrices $D\nu A^{-1}D\nu$ and A possessing the same property.

3) $D\nu B^{-1}D\nu = B$

4) $(B\tau, \tau) > 0$, $\tau \neq 0$, since the product of the positive-definite matrices $D\nu A^{-1}D\nu$ and A is a positive-definite matrix.

From the first property of matrix B it follows that all its eigenvalues are real. The second property signifies that Perron's theorem is applicable to matrix B, i.e. to its largest eigenvalue λ_m, being a simple root of the characteristic equation

$$|\lambda E - B| = 0 \tag{16}$$

there corresponds an eigenvector with positive coordinates. This vector is a solution of Eq. (14). Here $\beta = \alpha\lambda_m$. Properties 3 and 4 signify that matrix B has a spectrum symmetric relative to unity. This allows us, at least, to lower the order of Eq. (16) by not less than twice. For odd m the right-hand side of this equation contains the factor $(\lambda - 1)$ whose elimination reduces the order of the equation to an even one. For an Eq. (16) of even order a transition to the variable σ defined by the equality $\sigma = (\lambda^2 + 1)/2\lambda$ reduces the equation's order by two times.

Example 1. We consider the solution of Problem 1 for a vibrational system with two degrees of freedom. It should be noted that because of the conditions 2°– 4° imposed on the matrix of the generalized amplitude equation the matrix A of the vibrational system with two degrees of freedom is always oscillatory. Condition 5° is superfluous in this case since it follows from the preceding conditions. Here $\nu = (1, -1)$ (or $\nu = (-1, 1)$). Consequently,

$$B = \frac{1}{\Delta_-}\begin{Vmatrix} A_{22} & A_{12} \\ A_{21} & A_{11} \end{Vmatrix} \begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix}$$

$$|\lambda E - B| = \lambda^2 - 2\frac{\Delta_+}{\Delta_-}\lambda + 1, \quad \Delta_\pm = A_{11}A_{22} \pm A_{12}A_{21}$$

Solving characteristic Eq. (16), we obtain

$$\lambda_2 = \Delta_+^\circ / \Delta_-^\circ, \quad \Delta_\pm^\circ = \sqrt{A_{11}A_{22}} \pm \sqrt{A_{12}A_{21}}$$

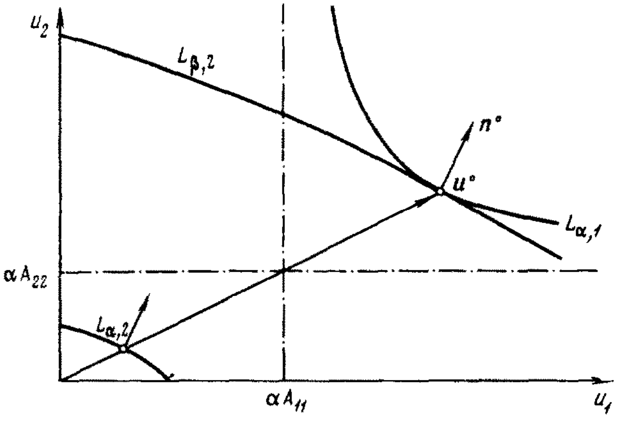

Fig. 1

Substituting the value of λ_2 into Eq. (15), we find vector z°

$$z^\circ = \sqrt{\rho}\,(1/\sqrt{A_{11}},\ 1/\sqrt{A_{22}})$$

It is convenient to select the value of the scalar factor ρ from the condition $(A\,z^\circ, z^\circ) = 1$

$$\rho = \sqrt{A_{11}A_{22}}\,/\,2\Delta_+^\circ$$

Then, the parameters of the vibrational system with a maximally compressed spectrum are determined by the relations

$$u^\circ = \frac{\alpha}{2\rho}(A_{11}, A_{22}), \quad \beta = \alpha\frac{\Delta_+^\circ}{\Delta_-^\circ},\ n^\circ = \rho\left(\frac{1}{A_{11}}, \frac{1}{A_{22}}\right)$$

The results obtained can be interpreted geometrically. Figure 1 shows the complete equifrequency surface L_α and the equifrequency surface $L_{\beta,2}$. L_α is the hyperbola

$$(u_1 - \alpha\, A_{11})\,(u_2 - \alpha\, A_{22}) = \alpha^2\, A_{12}\, A_{21}$$

The point u° is the point of tangency of surfaces $L_{\alpha,1}$ and $L_{\beta,2}$.

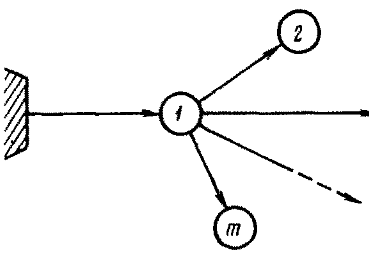

Fig. 2

Example 2. Let us consider a chain vibrational system with m degrees of freedom, whose structure is shown in Fig. 2. The elastic elements are denoted by arrows, the inertial ones by circles. The directions of the arrows determine the positive deformation of the corresponding elastic elements. An example of such a vibrational system is the mechanical model of a drive with the $(m-1)$-st working machine performing small torsional (longitudinal) oscillations.

Setting the square of the smallest natural frequency of the given vibrational system equal to α, we obtain the solution of Problem 1 by a variation of the rigidities of the elastic elements. We operate with the second inverse form of the amplitude equation, which satisfies all the requirements of Eq. (1). Here the quantities in Eq. (1) are defined thus: $\mathbf{u}$ is the vector of rigidities, $\mathbf{x}$ is the form of deformation of the elastic elements, μ is the square of the vibrational system's natural frequency

$$A = \begin{Vmatrix} J_0 & J_2 & J_3 & \ldots & J_m \\ J_2 & J_2 & 0 & \ldots & 0 \\ J_3 & 0 & J_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ J_m & 0 & 0 & \ldots & J_m \end{Vmatrix}, \quad \mathbf{J} = (J_1, J_2, \ldots J_m) \quad J_0 = \sum_{i=1}^{m} J_i$$

Here $\mathbf{J}$ is the vector of time-lags, J_0 is the total time-lag of the vibrational system. Matrix A is nonoscillatory when $m > 2$. According to [1], $\mathbf{v} = (1, -1, -1, \ldots, -1)$ and

$$D\mathbf{v}A^{-1}D\mathbf{v} = \begin{Vmatrix} h_1 & h_1 & h_1 & \ldots & h_1 \\ h_1 & h_1 + h_2 & h_1 & \ldots & h_1 \\ h_1 & h_1 & h_1 + h_3 & \ldots & h_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ h_1 & h_1 & h_1 & \ldots & h_1 + h_m \end{Vmatrix}$$

$$\mathbf{h} = (h_1, h_2, \ldots, h_m),\ h_i = J_i^{-1},\ i = 1, 2, \ldots, m$$

Here $\mathbf{h}$ is the vector of mobilities. Thus, setting $a = 2h_1J_0 - 1$, we obtain

$$B = \begin{Vmatrix} a & 2h_1J_2 & 2h_1J_3 & \ldots & 2h_1J_m \\ a+1 & 2h_1J_2+1 & 2h_1J_3 & \ldots & 2h_1J_m \\ a+1 & 2h_1J_2 & 2h_1J_3+1 & \ldots & 2h_1J_m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a+1 & 2h_1J_2 & 2h_1J_3 & \ldots & 2h_1J_m+1 \end{Vmatrix}$$

Having determined the roots of Eq. (16)

$$\lambda_1 = a - \sqrt{a^2-1}, \quad \lambda_2 = \lambda_3 = \cdots = \lambda_{m-1} = 1, \quad \lambda_m = a + \sqrt{a^2-1}$$

we substitute the value of λ_m into Eq. (15). From the set of its solutions we select

$$\mathbf{z}^\circ = \left(\sqrt{\frac{a-1}{a+1}}, 1, 1, \ldots, 1\right)$$

Using relation (7), we obtain the desired solution

$$u_1 = \alpha\,\mu_0\,J_0, \quad u_2 = \alpha\,\mu_0\,J_2, \quad u_3 = \alpha\,\mu_0\,J_3, \quad \ldots, \quad u_m = \alpha\,\mu_0\,J_m$$

Here

$$\mu_0 = \sqrt{\frac{a-1}{a+1}} + 1$$

The squares of the natural frequencies of the vibrational system obtained have the following values

$$\mu_1 = \alpha, \quad \mu_2 = \mu_3 = \ldots = \mu_{m-1} = \alpha\mu_0, \quad \mu_m = \alpha\,(a + \sqrt{a^2-1})$$

The method presented answers the question of how concentrated masses should be fixed on a weightless homogeneous freely-supported beam in order to have a compressed spectrum. For example, a vibrational system consisting of such a beam with three masses m_1, m_2 and m_3 positioned symmetrically has a compressed spectrum if

$$\frac{m_1}{m_2} - \frac{m_3}{m_2} = \frac{l^3}{8b^3(3l-4b)}$$

Here b is the distance of mass $m_1(m_3)$ from the nearest end of the beam and l is the beam's length.

REFERENCES

1. Mitin, V. N. and Shteinvol'f, L. I., Structure matrices of chain vibrational systems. In: Dynamics and Durability of Machines, № 17, Izd. Khar'kovsk. Univ., 1973.
2. Gantmakher, F. R. and Krein, M. G., Oscillatory Matrices and Kernels and Small Oscillations of Mechanical Systems, Moscow, Gostekhizdat, 1950.
3. Glazman, I. M. and Mitin, V. N., The tuning out of vibrational systems as a convex programing problem. Dokl. Akad. Nauk SSSR, Vol. 169, № 5, 1966.
4. Glazman, I. M. and Shteinvol'f, L. I., Elimination of hazardous-resonance zones of natural frequencies of a vibrational system by a variation of its parameters. Izv. Akad. Nauk SSSR, Mekhanika i Mashinostroenie, № 4, 1964.
5. Glazman, I. M. and Mitin, V. N., Optimal tuning out of torsional vibrational systems. In: Dynamics and Durability of Machines, № 6, Izd. Khar'kovsk. Univ., 1967.

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